

# Relativistic spin operator and Lorentz transformation of spin state of a massive Dirac particle

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We have shown the covariant relativistic spin operator is equivalent to the spin operator commuting with the free Dirac Hamiltonian. This implies that the covariant relativistic spin operator is a good quantum observable. The covariant relativistic spin operator has the pure quantum contribution which does not exist in the classical covariant spin operator. Based on this equivalence reduced spin states can be clearly defined. We have shown the change in the entropy of a reduced spin density matrix sweeps through the whole range according to the relative motion of an observer.

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## I. INTRODUCTION

The spin of a massive Dirac particle is  $1/2$  so it can be used as a qubit under the condition that a detector is ideal in the sense that it does not respond to the momentum of the particle. The statistical prediction of measurements for a spin is provided by the reduced spin density matrix which can be obtained by partial tracing over momentum degrees of freedom. The interesting fact is that the purity of the spin, which is described by the Von Neumann entropy of the reduced spin density matrix, changes under the Lorentz transformation [1]. This would be expected since the spin of a massive Dirac particle rotates depending on the momentum of the particle under the general Lorentz transformation as Wigner has noted [2]. By the Wigner rotation the entanglement of spin and momentum degrees of freedom can be changed in general.

The seminal work by Peres *et al.* [1] which noticed the relativistic depurification of the spin using two-component spinors has not been fully appreciated. There were arguments whether the reduced spin density matrix in Ref. [1] is a proper one [3]. This is partly because the physical spin observables was not noted in their paper [1]. The change of the spin entropy can be described by the reduced spin density matrix which does not require the explicit form of a spin observable. For the measurement of the spin entropy, however, the physical spin observables must be specified. Because of the lack of knowledge for the relativistic spin most of later works which are now known as the relativistic quantum information [4–17] have been done by using different kinds of spin than the one implicitly given in the paper [1].

Gingrich and Adami have used the 4-dimensional Dirac eigen-spinors which are spin part of the solution for the covariant Dirac equation, since it shows manifest Lorentz covariance [15]. The Dirac eigen-spinor in a laboratory

frame, where a particle is moving, has momentum dependent components so that the criticism for partial tracing over the momentum was given [7]. The momentum dependence in the components of the Dirac eigen-spinors can be eliminated in the special representation given by Foldy and Woutheyzen [18]. We call this as the Foldy-Woutheyzen (FW) representation. Therefore in the FW representation the meaning of partial tracing over the momentum is clear so that the reduced spin density matrix can be well defined. The two spin density matrices of the same positive energy particle in the FW representation and in the covariant Dirac representation give the same entropy. In this sense the results given by Peres *et al.* [1] and Gingrich *et al.* [15] can be justified. Recently it is noticed, however, that the reduced spin density matrix is meaningless since the spin of the relativistic particle cannot be measured independently of its momentum [16]. This conclusion is erroneous since they considered only the classical spin operator not the full quantum spin operator [19].

In the FW representation there is a mean spin operator which commutes with the free Dirac Hamiltonian so that it is a good quantum observable. We call this spin operator as the FW mean spin operator. Gürsey and Ryder [20] has shown that the covariant relativistic spin operator which is obtained by the Lorentz transformation in the spinor representation is the same as a FW mean spin operator for positive energy states. The equivalence for negative energy states, however, was not clear since the form of two operators are different. In this paper we will show explicitly the equivalence between the covariant relativistic spin operator and the FW mean spin operator for whole spinor space. The difference of the covariant relativistic spin operator from the classical covariant spin operator will also be discussed. This will clarify the long-standing controversy regarding what is the proper spin operator and the reduced spin entropy in a laboratory frame. We will also study the change of the relativistic spin entropy in the FW representation in which the tracing over the momentum has a clear meaning. The results show that the pure spin state can become the completely mixed spin state and vice versa. In section II and III we

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review the covariant relativistic spin operator and the FW mean spin for clarity and self-containedness. In section IV we will show the equivalence of two spin operators. In section V we will discuss the change of a reduced spin density matrix under Lorentz transformation. In section VI we will summarize our results.

## II. THE COVARIANT RELATIVISTIC SPIN

The spin operator in a laboratory frame can be obtained by the Lorentz transformation from the spin operator in the particle rest frame. This spin operator is a good candidate for a relativistic spin operator and we call this spin operator as the covariant relativistic spin operator. We will review the procedure to obtain the covariant relativistic spin in a fully covariant way and discuss the difference with the classical covariant spin used to define the covariant spin magnetic dipole moment. We also review another approach [21] to obtain the covariant relativistic spin operator using Pauli-Lubanski vector for completeness.

### A. Lorentz covariant approach

Let us consider the following covariant Dirac equation for a free massive Dirac particle in coordinate space,

$$(i\gamma^\mu \partial_\mu - m)\psi(p) = 0, \quad (1)$$

where

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3, \quad (2)$$

and  $\sigma_i$  are the usual Pauli matrices.  $\gamma^\mu$  are Dirac matrices. We use the sign convention  $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$  for the metric and the natural unit  $c = \hbar = 1$ . The summation conventions are also used. The Greek index  $\mu$  runs 0, 1, 2, 3 and the Latin index  $i$  runs 1, 2, 3. The covariant Dirac equation has two positive energy solutions  $u(p, s)e^{-ip_\mu x^\mu}$  and two negative energy solutions  $v(p, s)e^{ip_\mu x^\mu}$  where  $s = \pm 1$ .  $e^{-ip_\mu x^\mu}$  is the coordinate representation of the momentum eigenstate.  $u(p, s)$  and  $v(p, s)$  are spin parts of the Dirac solutions which are called a positive energy spinor and a negative energy spinor respectively. As is well-known, negative energy solutions can be interpreted as antiparticles with the opposite charge and the same momentum of the particles even if their momentum eigenvalue is  $-p_\mu = -(p_0, -\mathbf{p})$  [22]. In this paper we will use the term negative energy spinor since it is convenient for one-particle state.

Using the Lorentz covariance of the Dirac equation (1) the solutions for a moving particle can be obtained from the solutions in the particle rest frame by using the Lorentz transformation. The momentum of particle  $p_\mu$  is determined by the standard Lorentz boost  $L_{\mathbf{p}}$  from the momentum of the rest particle  $k_\mu = (m, 0, 0, 0)$

such as  $p_\mu = (L_{\mathbf{p}})_\mu^\nu k_\nu$ . Then the spinors  $u(p, s)$  and  $v(p, s)$  for moving particles can be obtained by multiplying  $\mathcal{S}(L_{\mathbf{p}})$  on the spinor  $u(k, s)$  and  $v(k, s)$  for the rest particle. That is,  $u(p, s) = \mathcal{S}(L_{\mathbf{p}})u(k, s)$  and  $v(p, s) = \mathcal{S}(L_{\mathbf{p}})v(k, s)$ . The  $\mathcal{S}(L_{\mathbf{p}})$  is the Lorentz transformation in the spinor representation corresponding to  $L_{\mathbf{p}}$

$$\begin{aligned} \mathcal{S}(L_{\mathbf{p}}) &= \begin{pmatrix} \cosh \frac{\xi}{2} & \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \sinh \frac{\xi}{2} \\ \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \sinh \frac{\xi}{2} & \cosh \frac{\xi}{2} \end{pmatrix}, \\ &= \frac{E + m - \gamma^0 \gamma^i p_i}{\sqrt{2m(E + m)}}, \end{aligned} \quad (3)$$

where  $\xi$  is the rapidity of the particle and  $\hat{\mathbf{p}} = \mathbf{p}/\sqrt{\mathbf{p}^2}$ . In the particle rest frame the meaning of the spin index  $s$  is clear since the spin operator in the particle rest frame is

$$\frac{\boldsymbol{\Sigma}}{2} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad (4)$$

where  $\boldsymbol{\Sigma}$  and  $\boldsymbol{\sigma}$  are three-dimensional vectors ( $\Sigma_x, \Sigma_y, \Sigma_z$ ) and  $(\sigma_x, \sigma_y, \sigma_z)$  respectively. The spinors  $u(k, \pm)$  and  $v(k, \pm)$  are eigen-spinors of the Pauli spin matrix  $\Sigma_z$  with eigenvalue  $\pm 1$ .

Note that the spinors  $u(p, s)$  and  $v(p, s)$  for moving particles have the same spin index  $s$  as the rest spinors  $u(k, s)$  and  $v(k, s)$ . This is reasonable since the Wigner rotation by a single standard Lorentz transformation becomes trivial [2]. It is not clear, however, what is the spin operator in a laboratory frame. The spin operator  $\frac{\boldsymbol{\Sigma}_i}{2}$  in Eq. (4) is  $-W_i/2m$  in the particle rest frame and  $W_i$  is the space part of the Pauli-Lubanski vector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\delta}J^{\nu\lambda}P^\delta, \quad (5)$$

where  $J^{\nu\lambda}$  and  $P^\delta$  are angular momentum and momentum operators.  $\epsilon_{\mu\nu\lambda\delta}$  is a Levi-Civita symbol. The  $W_i$  at a laboratory frame, however, does not satisfy the required the spin commutation relation  $[W_i, W_j] = -i\epsilon_{ijk}W_k/m$  although  $W_\mu W^\mu = -m^2 S(S+1)$  is the second Casimir invariant of the inhomogeneous Lorentz group (Poincaré group) and  $S$  is the spin of the particle [22].

The proper way to obtain the relativistic spin operator in a fully covariant way is to define the following spin tensor at the particle rest frame

$$\Sigma_0^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \quad (6)$$

In the standard representation the 3-dimensional spin vector defined by  $\Sigma_i = \epsilon_{ijk}\Sigma_0^{jk}/2$  becomes  $4 \times 4$  Pauli spin  $\Sigma_i$ , where  $\epsilon_{ijk}$  is a Levi-Civita symbol. The spin operator at a moving frame can naturally be defined by the Lorentz transformation from the spin tensor at the particle rest frame as

$$\Sigma_{\mathbf{R}}^{\mu\nu} = \mathcal{S}(L_{\mathbf{p}})\Sigma_0^{\mu\nu}\mathcal{S}^{-1}(L_{\mathbf{p}}), \quad (7)$$

where  $\mathcal{S}^{-1}(L_{\mathbf{p}})$  is the inverse Lorentz transformation of  $\mathcal{S}(L_{\mathbf{p}})$ . Then the spin operator becomes

$$\begin{aligned} (\Sigma_R)_i &= \frac{\epsilon_{ijk} \Sigma_R^{jk}}{2} \\ &= \frac{\Sigma_i}{2} + \frac{\epsilon_{ijk} p_j (\boldsymbol{\Sigma} \times \mathbf{p})_k}{2m(E+m)} + i \frac{\gamma_5}{2m} (\boldsymbol{\Sigma} \times \mathbf{p})_i, \end{aligned} \quad (8)$$

which is the same as the covariant relativistic spin operator in Ref. [20]. The  $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$  is a  $4 \times 4$  matrix with off-diagonal terms. Notice that the eigenvalue equations of the covariant relativistic spin operator in the laboratory frame are

$$\begin{aligned} (\Sigma_R)_z u(p, \pm) &= \pm u(p, \pm), \\ (\Sigma_R)_z v(p, \pm) &= \pm v(p, \pm). \end{aligned} \quad (9)$$

These confirm that the spin eigenvalues in the laboratory frame is the same as the spin eigenvalues in the particle rest frame. An arbitrary positive energy spin state  $\psi(k) = au(k, +) + bu(k, -)$  in the particle rest frame becomes the spin state  $\psi(p) = \mathcal{S}(L_{\mathbf{p}})\psi(k, s) = au(p, +) + bu(p, -)$  in the moving frame. Then the expectation value of the spin in the particle rest frame  $\bar{\psi}(k)\boldsymbol{\Sigma}\psi(k)$  is equal to the expectation value of the spin in the moving frame  $\bar{\psi}(p)\boldsymbol{\Sigma}_R\psi(p)$ . Where  $\bar{\psi}(p) = \psi(p)^\dagger \gamma_0$  and the relativistic invariant normalizations

$$\begin{aligned} \bar{u}(p, s)u(p, s') &= -\bar{v}(p, s)v(p, s') = \delta_{ss'}, \\ \bar{u}(p, s)v(p, s') &= 0 \end{aligned} \quad (10)$$

are used. This implies that the measurements of the spin along the same axes by two observers, one in the particle rest frame and the other in the laboratory frame, will give the same spin expectation values.

It is interesting to compare the covariant relativistic spin operator in Eq. (8) with the classical covariant spin in classical electrodynamics. The classical covariant spin can be defined as  $\mathbf{S} = \gamma_v \hat{\boldsymbol{\mu}}/\alpha$  by using the classical covariant magnetic dipole moment [23]

$$\hat{\boldsymbol{\mu}} = \alpha \left[ \frac{\boldsymbol{\sigma}}{2} - \frac{\mathbf{p}(\mathbf{v} \cdot \boldsymbol{\sigma})}{2(E+m)} \right] \quad (11)$$

Where  $\alpha$  is the gyromagnetic ratio. The Lorentz factor  $\gamma_v$  is  $1/\sqrt{1-\mathbf{v}^2}$  and  $\mathbf{p} = m\gamma_v \mathbf{v}$ . Notice that the classical covariant spin  $\mathbf{S}$  can also be derived by the spin tensor  $S_{\mu\nu} = -\epsilon_{\mu\nu\lambda\delta} P^\lambda W^\delta$ . One can see the main difference of the covariant relativistic spin operator  $\boldsymbol{\Sigma}_R$  and the classical spin operator  $\mathbf{S}$  is the  $\gamma_5$  proportional term. That is, without  $\gamma_5$  proportional term the covariant relativistic spin operator  $\boldsymbol{\Sigma}_R$  becomes  $\mathbf{S}\mathcal{I}$ , where  $\mathcal{I}$  is the  $4 \times 4$  identity matrix. Therefore the  $\gamma_5$  proportional term is purely quantum mechanical which is manifest in the 4-dimensional spinor representation.

### B. Construction by using commutators of Pauli-Lubanski vectors

The Dirac particles satisfy the inhomogeneous Lorentz group (Poincaré group) symmetry. There are two

Casimir operators in the Poincaré group, the first Casimir invariant  $P^\mu P_\mu$  and the second Casimir invariant  $W^\mu W_\mu$ . So it was expected that the covariant relativistic spin operator can be constructed directly by the Pauli-Lubanski vector. The covariant relativistic spin operator, however, cannot be obtained by a simple linear combinations of the components of the Pauli-Lubanski vector. It was shown that the only candidate for the components of angular momentum vector which is a linear function of  $W^\mu$  is [24]

$$\frac{1}{M} \left( W_i - \frac{P_i W^0}{m+E} \right). \quad (12)$$

In fact the above equation is the spin operator  $\Sigma_i$  in the particle rest frame which is written by  $W_\mu$  in a laboratory frame.

Ryder has shown the correct way to construct the spin operator directly from the Pauli-Lubanski vector other than the usual Pauli spin operator [21]. First the following two antisymmetric tensor operators are defined by commutators of Pauli-Lubanski vectors,

$$W^{\mu\nu} = \frac{1}{m^2} [W^\mu, W^\nu], \quad \tilde{W}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\delta} W_{\rho\delta}. \quad (13)$$

Then the two tensor operators,

$$X^{\mu\nu} = -i \left( W^{\mu\nu} + i\tilde{W}^{\mu\nu} \right), \quad (14)$$

$$Y^{\mu\nu} = -i \left( W^{\mu\nu} - i\tilde{W}^{\mu\nu} \right), \quad (15)$$

satisfy the commutation relations for the angular momentum. Therefore these two tensor operators can be represented by  $2 \times 2$  matrices which cannot be irreducible representations under the parity. The 4-dimensional irreducible spin tensor including parity can be constructed as

$$(\Sigma_R)_{\mu\nu} = \frac{1}{2} (1 - \gamma_5) X_{\mu\nu} + \frac{1}{2} (1 + \gamma_5) Y_{\mu\nu}. \quad (16)$$

Then the spatial component defined by  $(\Sigma_R)_i = \frac{1}{2} \epsilon_{ijk} (\Sigma_R)_{jk}$  becomes the same as the covariant relativistic spin operator in Eq. (8).

### III. CANONICAL APPROACH

In this section we will review the canonical approach given by Foldy and Woutheyse [18]. The covariant relativistic spin operator was obtained in the last section, however, it is not clear whether the covariant relativistic spin operator is a good quantum observable in the sense that good quantum observables must commute with the Hamiltonian of a particle. In the canonical approach the following free Dirac Hamiltonian is used to describe a massive Dirac particle in a laboratory frame,

$$\mathcal{H}_D(\mathcal{P}) = \gamma^0 m - \gamma^0 \gamma^i \mathcal{P}_i. \quad (17)$$

The  $\mathcal{P}_i$  is a momentum operator such that the eigenvalues for a positive energy state and a negative energy state are  $\pm p_i$  respectively. In the particle rest frame the covariant relativistic spin operator and the free Dirac Hamiltonian commute each other. In a laboratory frame, however, the covariant relativistic spin operator  $\Sigma_R$  does not commute with the free Dirac Hamiltonian in Eq. (17). This is because the Dirac Hamiltonian in a laboratory frame cannot be obtained simply by the Lorentz transformation from the Dirac Hamiltonian at the particle rest frame as the covariant relativistic spin operator.

Foldy and Wouthuysen (FW) have found the mean spin operator which commutes with the Dirac Hamiltonian for a moving particle by using the following canonical transformation [18]

$$\mathcal{U}(\mathcal{P}) = \frac{E + m - \gamma^i \mathcal{P}_i}{\sqrt{2E(E + m)}}. \quad (18)$$

In the new representation the Dirac Hamiltonian becomes the diagonal form such as  $\tilde{\mathcal{H}}_D(\mathcal{P}) = \mathcal{U}(\mathcal{P})\mathcal{H}_D(\mathcal{P})\mathcal{U}^\dagger(\mathcal{P}) = \gamma^0 E_{\mathbf{p}}$ , where  $E_{\mathbf{p}}$  represents the operator which gives eigenvalues  $\pm E = \pm\sqrt{\mathbf{p}^2 + m^2}$ . We call this new representation as the FW representation. In the FW representation a spinor transforms as

$$\tilde{\psi}^\pm(p) = \sqrt{\frac{m}{E}} \mathcal{U}(\pm\mathbf{p}) \psi^\pm(p). \quad (19)$$

Where  $\psi^\pm(p)$  are the positive energy and the negative energy Dirac spinors in the standard representation such that  $\mathcal{P}\psi^\pm(p) = \pm\mathbf{p}\psi^\pm(p)$ . We use  $\tilde{\sim}$  symbol for the objects in the FW representation. Here the factor  $\sqrt{\frac{m}{E}}$  is required because of the different normalizations in the canonical representation and in the covariant representation. In the canonical representation  $\tilde{\psi}^\pm(p)^\dagger \tilde{\psi}^\pm(p) = 1$  normalization is used and  $\psi^\pm(p)^\dagger \gamma^0 \psi^\pm(p) = \pm 1$  normalization is used in the covariant representation.

Notice that the Dirac Hamiltonian of the moving particle in the FW representation is the diagonal form  $\gamma^0 E_{\mathbf{p}}$ . Therefore the  $4 \times 4$  Pauli spin operator  $\Sigma$  commutes with the transformed Hamiltonian  $\tilde{\mathcal{H}}_D(\mathcal{P})$ , hence the spin operator  $\Sigma$  becomes a good spin observable in the FW representation. The spin operator  $\tilde{\Sigma}$  for a moving particle in the FW representation, which is  $\Sigma$ , has the following positive and negative energy eigen-spinors

$$\begin{aligned} \tilde{u}(p, +) &= (1, 0, 0, 0)^T, & \tilde{u}(p, -) &= (0, 1, 0, 0)^T, \\ \tilde{v}(p, +) &= (0, 0, 1, 0)^T, & \tilde{v}(p, -) &= (0, 0, 0, 1)^T, \end{aligned} \quad (20)$$

where the superscript T means the transpose of a vector.

The spin operator  $\tilde{\Sigma}$  in the FW representation is transformed to the FW mean spin operator  $\Sigma_{FW}$  in the standard representation as

$$\begin{aligned} \frac{\Sigma_{FW}}{2} &= \mathcal{U}^\dagger(\mathcal{P}) \frac{\tilde{\Sigma}}{2} \mathcal{U}(\mathcal{P}) \\ &= \frac{\Sigma}{2} - \frac{i\gamma^0(\Sigma \times \mathcal{P})}{2E} - \frac{\mathcal{P} \times (\Sigma \times \mathcal{P})}{2E(E + m)}. \end{aligned} \quad (21)$$

One can easily check the FW spin operator  $\Sigma_{FW}/2$  commutes with the Dirac Hamiltonian  $\mathcal{H}_D(\mathcal{P})$  by direct calculation. This means the FW mean spin operator is a constant of motion and becomes a good quantum observable in the standard representation. As mention in Ref. [18] the spin operator  $\Sigma_{FW}$  is nonlocal since it depends on the momentum. In fact the eigenvalue of the FW mean spin operator represents the average spin of rapidly oscillating particle within its Compton wavelength. This is the meaning of the name the FW mean spin operator. The FW mean spin operator becomes the spin operator in the non-relativistic Pauli Hamiltonian. This implies that the expectation value measured by the experiment in the non-relativistic quantum mechanics is the expectation value of this spin operator.

#### IV. EQUIVALENCE BETWEEN RELATIVISTIC SPIN AND FW MEAN SPIN

The covariant relativistic spin has been shown to be equivalent to the Foldy-Wouthuysen spin for the positive energy states by several authors [20]. For the positive energy spinors the spinor representation  $\mathcal{S}(L_{\mathbf{p}})$  of the standard Lorentz transformation  $L_{\mathbf{p}}$  in Eq. (3) can be represented by the unitary operator  $\mathcal{U}(\mathcal{P})$  in Eq. (18) with the normalization factor  $\sqrt{\frac{m}{E}}$  so the equivalence of the two spin operators can be given in the operator level. The spinor representation of the Lorentz transformation acting on the entire spinor space, however, cannot be a unitary operator since the Lorentz group has no finite dimensional unitary representation. At this stage, therefore, it is unclear that the covariant relativistic spin operator is equivalent to the FW mean spin operator for negative energy states. In fact the two spin operators, the covariant relativistic spin  $\Sigma_R$  in Eq. (8) and the FW spin  $\Sigma_{FW}$  in Eq. (21) are different. It raises the question which spin operator is the proper spin observable for the massive Dirac particles. We will show that they are equivalent in the sense that they give the same expectation values.

The equivalence of the two spin operators can be obtained by using the relations between the actions of the Lorentz transformation  $\mathcal{S}(L_{\mathbf{p}})$  and the unitary transformation  $\mathcal{U}(\mathcal{P})$  on the positive and negative energy spinors. The unitary operator  $\mathcal{U}(\mathcal{P})$  has the momentum operator  $\mathcal{P}$  which gives different eigenvalues between positive energy and negative energy spinors. So it is convenient to use the projection operators  $\Pi = (m + \gamma^\mu \mathcal{P}_\mu)/2$  such that

$$\Pi u(p, s) = \frac{m + \gamma^\mu p_\mu}{2m} u(p, s) = u(p, s), \quad (22)$$

$$\Pi v(p, s) = \frac{m - \gamma^\mu p_\mu}{2m} v(p, s) = v(p, s). \quad (23)$$

One can define  $\Pi_+ = (m + \gamma^\mu p_\mu)/2$  and  $\Pi_- = (m - \gamma^\mu p_\mu)/2$  which satisfy  $\Pi_+^2 = \Pi_+$ ,  $\Pi_-^2 = \Pi_-$  and  $\Pi_+ \Pi_- = \Pi_- \Pi_+ = 0$ . That is,  $\Pi_+$  and  $\Pi_-$  project out positive and negative energy spinors respectively. Then the following



relations hold

$$\mathcal{U}(\mathbf{p})\Pi_+ = \sqrt{\frac{E}{m}}\mathcal{S}^{-1}(L_{\mathbf{p}})\Pi_+, \quad (24)$$

$$\mathcal{U}(-\mathbf{p})\Pi_- = \sqrt{\frac{E}{m}}\mathcal{S}^{-1}(L_{\mathbf{p}})\Pi_-,$$

$$\Pi_+^\dagger \mathcal{U}^\dagger(\mathbf{p}) = \sqrt{\frac{E}{m}}\Pi_+^\dagger \gamma^0 \mathcal{S}(L_{\mathbf{p}}),$$

$$\Pi_-^\dagger \mathcal{U}^\dagger(-\mathbf{p}) = -\sqrt{\frac{E}{m}}\Pi_-^\dagger \gamma^0 \mathcal{S}(L_{\mathbf{p}}). \quad (25)$$

Using these relations we obtain the following equivalences

$$\begin{aligned} \frac{m}{E} \tilde{u}^\dagger(p, s) \tilde{\Sigma} \tilde{u}(p, s) &= \frac{m}{E} u^\dagger(p, s) \Sigma_{\text{FW}} u(p, s) \\ &= \tilde{u}(p, s) | \Sigma_{\text{R}} u(p, s), \end{aligned} \quad (26)$$

and

$$\begin{aligned} \frac{m}{E} \tilde{v}^\dagger(p, s) \tilde{\Sigma} \tilde{v}(p, s) &= \frac{m}{E} v^\dagger(p, s) \Sigma_{\text{FW}} v(p, s) \\ &= -\tilde{v}(p, s) | \Sigma_{\text{R}} v(p, s), \end{aligned} \quad (27)$$

since  $\Pi_+ \tilde{u}(p, s) = \tilde{u}(p, s)$  and  $\Pi_- \tilde{v}(p, s) = \tilde{v}(p, s)$ . The  $-$  sign and  $m/E$  are due to the relativistic invariant normalization in Eq. (10).

Eqs. (26) and (27) guarantee that the expectation values of the FW mean spin operator and the covariant relativistic spin operator are the same for the general spinor  $\psi(p)$  which is a linear combination of the positive and negative eigen-spinors. In this sense the two spin operators are equivalent. As a result the covariant relativistic spin operator can be considered as a good quantum observable since it is equivalent to the FW mean spin operator which commutes with the free Dirac Hamiltonian.

## V. REDUCED SPIN DENSITY OPERATOR

To study a reduced spin density operator the momentum representation of the covariant Dirac equation is convenient. The solutions of the covariant Dirac equation in the momentum representation can be represented as  $|p\rangle \otimes u(p, s)$  for positive energy solutions and  $|-p\rangle \otimes v(p, s)$  for negative energy solutions. Where  $|p\rangle$  is the momentum eigenstate, i.e.,  $P_\mu |p\rangle = p_\mu |p\rangle$ . The spin expectation value of a spinor in a laboratory frame has been shown to be the same as the spin expectation value of the spinor in the particle rest frame. This implies that a reduced spin density matrix can be clearly defined by the partial tracing over momentum degrees of freedom. To make this point more transparent we will study reduced spin density matrices in the FW representation since the spin and momentum degrees of freedoms are decoupled in the FW representation as those in the particle rest frame. In the FW representation the meaning of the partial trace over the momentum is clear.

We consider the following state in the FW representation,

$$\begin{aligned} \tilde{\psi} &= \int d^3\mathbf{p} \tilde{\psi}(p) \otimes |p\rangle \\ &= \int d^3\mathbf{p} \sum_{\delta} [a_{\delta}(\mathbf{p}) \tilde{u}(p, \delta) + b_{\delta}(\mathbf{p}) \tilde{v}(p, \delta)] \otimes |p\rangle. \end{aligned} \quad (28)$$

The normalization  $\tilde{\psi}^\dagger \tilde{\psi} = 1$  requires  $\int d^3\mathbf{p} \sum_{\delta} [a_{\delta}(\mathbf{p}) a_{\delta}^*(\mathbf{p}) + b_{\delta}(\mathbf{p}) b_{\delta}^*(\mathbf{p})] = 1$ . For the density matrix  $\tilde{\rho} = \tilde{\psi} \tilde{\psi}^\dagger$ , the reduced spin density matrix is well defined by the partial trace over momentum as

$$\begin{aligned} \tilde{\rho}_s &= \text{Tr}_{\mathbf{p}} \tilde{\rho} = \int d^3\mathbf{p} \tilde{\psi}^\dagger(p) \tilde{\psi}(p) \\ &= \int d^3\mathbf{p} \sum_{\lambda, \lambda'} [a_{\lambda}(\mathbf{p}) a_{\lambda'}^*(\mathbf{p}) \tilde{u}(p, \lambda) \tilde{u}^\dagger(p, \lambda') \\ &\quad + a_{\lambda}(\mathbf{p}) b_{\lambda'}^*(\mathbf{p}) \tilde{u}(p, \lambda) \tilde{v}^\dagger(p, \lambda') \\ &\quad + b_{\lambda}(\mathbf{p}) a_{\lambda'}^*(\mathbf{p}) \tilde{v}(p, \lambda) \tilde{u}^\dagger(p, \lambda') \\ &\quad + b_{\lambda}(\mathbf{p}) b_{\lambda'}^*(\mathbf{p}) \tilde{v}(p, \lambda) \tilde{v}^\dagger(p, \lambda')] \end{aligned} \quad (29)$$

The reduced spin density matrix becomes the same as the non-relativistic spin density matrix for a positive energy particle.

The relativistic effects on the reduced spin density matrix of a moving particle is represented by the transformation of the reduced spin density matrix under a general Lorentz transformation. The transformation of the reduced spin density matrix is given by the transformation matrix of the spinor under the Lorentz transformation. We have obtained the transformation matrix for the positive energy spinors in the previous work [17]. In this section the complete transformation matrix for the whole spinor space is obtained. The complete transformation matrix in the FW representation will be shown to be equivalent to the complete transformation matrix in the covariant representation.

Let us assume that the two observers  $\mathcal{O}$  and  $\mathcal{O}'$  are related by an arbitrary Lorentz transformation  $\Lambda$ . Lorentz transformations do not change the sign of the energy for a free particle so that we deal the positive and negative energy states separately. The transformation matrix  $\mathcal{T}^{(\Lambda, \mathbf{p})}$  in the covariant representation and the transformation matrix  $\tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})}$  in the FW representation for the positive energy spinors have the following relations

$$\begin{aligned} \mathcal{S}(\Lambda) u(p, \lambda) &= \mathcal{S}(\Lambda) \mathcal{S}(L_{\mathbf{p}}) u(k, \lambda) \\ &= \sum_{\lambda'} \mathcal{T}_{\lambda' \lambda}^{(\Lambda, \mathbf{p})} \mathcal{S}(L_{\Lambda \mathbf{p}}) u(k, \lambda') \\ &= \sqrt{\frac{E}{m}} \mathcal{S}(\Lambda) \mathcal{U}^\dagger(\mathbf{p}) \tilde{u}(p, \lambda) \\ &= \sqrt{\frac{E}{m}} \sum_{\lambda'} \tilde{\mathcal{T}}_{\lambda' \lambda}^{(\Lambda, \mathbf{p})} \mathcal{U}^\dagger(\mathbf{p}) \tilde{u}(\Lambda p, \lambda'), \end{aligned} \quad (30)$$

where  $\Lambda \mathbf{p}$  is the space part of the four momentum  $(\Lambda p)_\mu = \Lambda_\mu^\nu p_\nu$ . This equation shows that the transformation matrix  $\mathcal{T}_{\lambda'\lambda}^{(\Lambda, \mathbf{p})} = u(k, \lambda') \mathcal{S}^{-1}(L_{\Lambda \mathbf{p}}) \mathcal{S}(\Lambda) \mathcal{S}(L_{\mathbf{p}}) u(k, \lambda)$  in the covariant representation is equivalent to the transformation matrix  $\tilde{\mathcal{T}}_{\lambda'\lambda}^{(\Lambda, \mathbf{p})} = \tilde{u}(\Lambda p, \lambda') \mathcal{U}(\mathbf{P}) \mathcal{S}(\Lambda) \mathcal{U}^\dagger(\mathbf{P}) \tilde{u}(p, \lambda)$  in the FW representation since the forms of  $u(k, \lambda)$  and  $\tilde{u}(p, \lambda)$  are the same. The transformation matrices for the negative energy spinor is obtained in a similar manner. As a result, the complete transformation matrix can effectively be written as the following block diagonal form

$$\tilde{\mathcal{T}}^{\Lambda, \mathbf{p}} = \begin{pmatrix} A & B & 0 & 0 \\ -B^* & A & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -B^* & A \end{pmatrix}, \quad (31)$$

where  $|A|^2 + |B|^2 = 1$  and  $A^*B - AB = 0$ . This transformation matrix shows that the irreducible representation for the Lorentz transformation  $\Lambda$  is the  $2 \times 2$  unitary matrix  $\begin{pmatrix} A & B \\ -B^* & A \end{pmatrix}$ . This two-dimensional matrix corresponds to the two-dimensional unitary representation of the Wigner rotation. This means the FW representation and the covariant representation are two equivalent representations for the Wigner's little group.

Now we will consider the change of the reduced spin density matrix  $\tilde{\rho}_s$  under the Lorentz transformation  $\mathcal{S}(\Lambda)$ . The transformation of the spinor  $\tilde{u}(p)$  is obtained as

$$\begin{aligned} \tilde{\psi}(\Lambda p) &= \mathcal{U}(\mathbf{P}) \mathcal{S}(\Lambda) \mathcal{U}^\dagger(\mathbf{P}) \tilde{\psi}(p) \\ &= \sum_{\lambda', \lambda} \tilde{\mathcal{T}}_{\lambda'\lambda}^{\Lambda, \mathbf{p}} (a_\lambda(\mathbf{p}) \tilde{u}(\Lambda p, \lambda') + b_\lambda(\mathbf{p}) \tilde{v}(\Lambda p, \lambda')) \end{aligned} \quad (32)$$

The reduced spin density matrix for the observer  $\mathcal{O}'$ , defined by

$$\tilde{\rho}'_s = \int d^3 \mathbf{p} \tilde{\psi}(\Lambda p) \tilde{\psi}(\Lambda p), \quad (33)$$

shows the rotation of the spin depending not only on the Lorentz transformation  $\Lambda$  but also the momentum of the particle. This momentum dependent rotation changes the entanglement between the spin and the momentum degrees of freedom. As a result the spin entropy which describes the mixedness of the reduced spin density matrix changes.

The reduced spin density matrix for the general state  $\tilde{\psi}$  in Eq. (28) is too complicated to study the essential feature of the spin density matrix under the general Lorentz transformation. Therefore we will consider the simple example which shows the nontrivial effects under the Lorentz transformation. We consider the following two initial positive energy states in the FW representa-

tion,

$$\tilde{\psi}_1 = \frac{1}{\sqrt{2\delta(\mathbf{0})}} (\tilde{u}(p, +) \otimes |p\rangle + \tilde{u}(p_\perp, +) \otimes |p_\perp\rangle), \quad (34)$$

$$\tilde{\psi}_2 = \frac{1}{\sqrt{2\delta(\mathbf{0})}} (\tilde{u}(p, +) \otimes |p\rangle + \tilde{u}(p_\perp, -) \otimes |p_\perp\rangle), \quad (35)$$

where the four momentum  $p^\mu = (E, \mathbf{p})$  and  $p_\perp^\mu = (E, \mathbf{p}_\perp)$ .  $1/\sqrt{2\delta(\mathbf{0})}$  is included due to the normalization  $\tilde{\psi}_1^\dagger \tilde{\psi}_1 = \tilde{\psi}_2^\dagger \tilde{\psi}_2 = 1$ . The space parts of the momentum are written in the spherical coordinate as  $\mathbf{p} = \{p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta\}$  and  $\mathbf{p}_\perp = (E, p \cos \theta \cos \phi, p \cos \theta \sin \phi, -p \sin \theta)$ .  $\mathbf{p}_\perp$  is perpendicular to  $\mathbf{p}$  and corresponds to the replacement of  $\theta$  by  $\theta + \pi/2$ . The  $\theta$  is the polar angle from the positive  $z$ -axis and  $\phi$  is the azimuthal angle in the  $xy$  plane from the  $x$ -axis. The action of a rotation is trivial since a rotation does not change the entanglement between the momentum degrees of freedom and the spin degrees of freedom. Therefore we consider the Lorentz boost  $\Lambda_\xi$  in  $z$  direction with rapidity  $\xi$ . This does not reduce the generality since the general momentum vector  $p^\mu$  is considered. The reduced spin density matrices can be described by  $2 \times 2$  matrices since the positive and negative energy spinors in the FW representation can be represented by two components vectors. The reduced spin density matrices  $\tilde{\rho}_{1s}$  and  $\tilde{\rho}_{2s}$  are obtained by tracing over the momentum for the density matrices  $\tilde{\rho}_1 = \tilde{\psi}_1 \tilde{\psi}_1^\dagger$  and  $\tilde{\rho}_2 = \tilde{\psi}_2 \tilde{\psi}_2^\dagger$  respectively. The  $\tilde{\rho}_{1s}$  becomes the pure state  $\tilde{u}(p, +) \tilde{u}^\dagger(p, +)$  and the  $\tilde{\rho}_{2s}$  becomes the complete mixed state  $[\tilde{u}(p, +) \tilde{u}^\dagger(p, +) + \tilde{u}(p_\perp, -) \tilde{u}^\dagger(p_\perp, -)]/2$ . Note that the two-dimensional representation of the spinor  $\tilde{u}(p, \pm)$  in the FW representation is described by  $|\pm\rangle$  which are the eigenstates of the usual Pauli matrix  $\sigma_z$  with eigenvalues  $\pm 1$  such as  $\sigma_z |\pm\rangle = |\pm\rangle$ .

The transformed spin density matrices become

$$\begin{aligned} \tilde{\rho}'_{1s} &= \frac{1}{2} \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})} |+\rangle \langle +| \left( \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})} \right)^\dagger \\ &+ \frac{1}{2} \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p}_\perp)} |+\rangle \langle +| \left( \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p}_\perp)} \right)^\dagger \\ &= \frac{1}{2} \begin{pmatrix} a_1^2 + a_2^2 & -(a_1 b_1 + a_2 b_2) e^{-i\phi} \\ -(a_1 b_1 + a_2 b_2) e^{i\phi} & b_1^2 + b_2^2 \end{pmatrix}. \end{aligned} \quad (36)$$

and

$$\begin{aligned} \tilde{\rho}'_{2s} &= \frac{1}{2} \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})} |+\rangle \langle +| \left( \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})} \right)^\dagger \\ &+ \frac{1}{2} \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p}_\perp)} |-\rangle \langle -| \left( \tilde{\mathcal{T}}^{(\Lambda, \mathbf{p}_\perp)} \right)^\dagger \\ &= \frac{1}{2} \begin{pmatrix} a_1^2 + b_2^2 & (a_2 b_2 - a_1 b_1) e^{-i\phi} \\ (a_2 b_2 - a_1 b_1) e^{i\phi} & b_1^2 + a_2^2 \end{pmatrix}. \end{aligned} \quad (37)$$

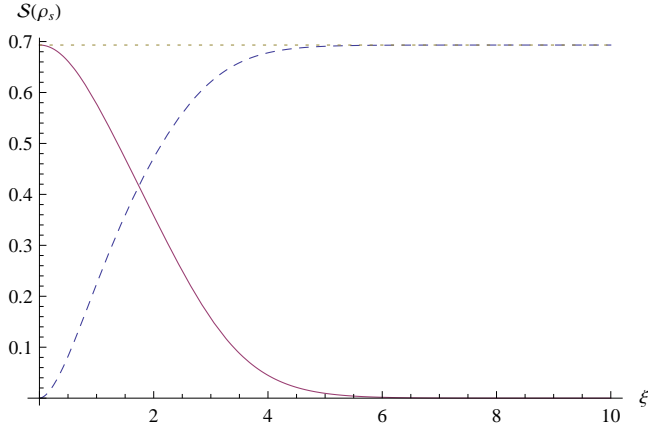


FIG. 1: (Color on-line) This figure shows the change of entropies of the spin density matrices  $\tilde{\rho}'_{1s}$  (blue dashed line) and  $\tilde{\rho}'_{2s}$  (red real line) as a function of rapidity with  $\sqrt{\mathbf{p}^2} = 10$  and  $m = 1$  for  $\theta = 0.54\pi$ . The brown dotted line represents  $S = \ln 2$  corresponding to the completely mixed state.

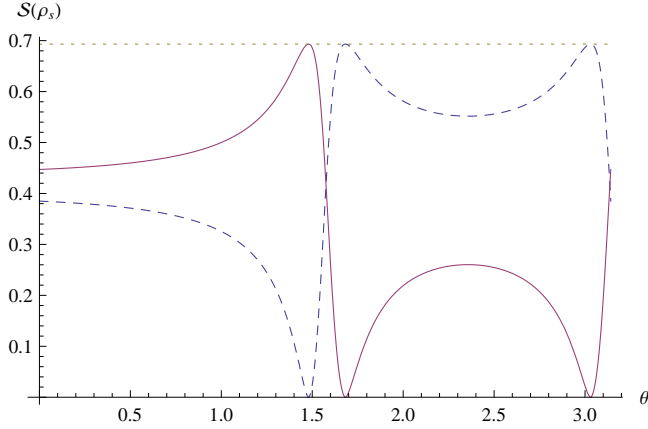


FIG. 2: (Color on-line) This figure shows the behavior of the spin entropies as functions of the polar angle  $\theta$  for  $\tilde{\rho}'_{1s}$  (blue dashed line) and  $\tilde{\rho}'_{2s}$  (red real line) with  $\sqrt{\mathbf{p}^2} = 10$  and  $m = 1$  for  $\xi = 10$ . The brown dotted line represents  $S = \ln 2$  of the completely mixed state.

The parameters  $a_1$ ,  $a_2$ ,  $b_1$  and  $b_2$  are

$$a_1 = \sqrt{\frac{m+E}{m+E'}} \left[ \cosh \frac{\xi}{2} + \frac{p \cos \theta}{m+E} \sinh \frac{\xi}{2} \right], \quad (38)$$

$$b_1 = \frac{p \sin \theta}{\sqrt{(m+E)(m+E')}} \sinh \frac{\xi}{2},$$

$$a_2 = \sqrt{\frac{m+E}{m+E''}} \left[ \cosh \frac{\xi}{2} + \frac{-p \sin \theta}{m+E} \sinh \frac{\xi}{2} \right], \quad (39)$$

$$b_2 = \frac{p \cos \theta}{\sqrt{(m+E)(m+E'')}} \sinh \frac{\xi}{2}, \quad (40)$$

where  $E' = E \cosh \xi + p \cos \theta \sinh \xi$  and  $E'' = E \cosh \xi - p \sin \theta \sinh \xi$ .

We will study the change of the purification of the spin by the entropy of the spin density matrix. The entropy

for the state  $\rho_s$  is defined as [25]

$$S = -\text{Tr}(\rho_s \ln \rho_s) = -\sum_j \lambda_j \ln \lambda_j, \quad (41)$$

where  $\lambda_j$  are the eigenvalues of  $\rho_s$ . The spin density matrices transform nontrivially under Lorentz transformation as shown in Eq. (36) and Eq. (37) so that the spin entropies will also change nontrivially. The results are shown in Fig. 1 and Fig. 2. For explicit calculations we have set the momentum  $\sqrt{\mathbf{p}^2} = 10$  and the rest mass  $m = 1$ . Fig. 1 shows the dependence on the rapidity  $\xi$  for  $\theta = 0.54\pi$ . The magnitude of the observer's velocity becomes  $0.9999999959c$  for  $\xi = 10$ . The changes of spin entropies in this figure show that the pure state can be changed to the complete mixed state and vice versa, according to the increase of the rapidity. Fig. 2 shows the polar angle dependence of the spin entropies for the rapidity  $\xi = 10$ . The spin entropies in this figure show that the Lorentz transformation changes mixedness of the spin through the whole range depending on the polar angle.

According to the fact that the transformation matrices  $\mathcal{T}^{(\Lambda, \mathbf{p})}$  and  $\tilde{\mathcal{T}}^{(\Lambda, \mathbf{p})}$  in the covariant representation and in the FW representation are equivalent, the behaviors of the reduced spin density matrix in the two representations are the same. The difference in the two representations are the normalizations. In the covariant representation the density matrix for the observer  $\mathcal{O}$  is defined as  $\rho = \psi \psi^\dagger \gamma^0$  to guarantee the Lorentz invariance of the normalization. The trace over momentum is also represented by the Lorentz invariant measure  $d^3\mathbf{p}/2E$ .

## VI. SUMMARY

In summary, we have shown the equivalence between the covariant relativistic spin and the FW mean spin of a Dirac particle. Based on the equivalence it is clear that the covariant relativistic spin operator is a good spin operator in the covariant representation. As a result, it can be understood that the spin index of the Dirac spinor represents the spin eigenvalues of the moving particle. The covariant relativistic spin is shown to have a pure quantum contribution which cannot be given by the classical spin. In the FW representation the Dirac Hamiltonian for a moving particle becomes the diagonal form as the Hamiltonian in the particle rest frame. This fact makes it easy to deal the momentum and spin degrees of freedom separately. We have studied the relativistic effects on the spin state in the FW representation. The spin state can be defined by tracing over momentum degrees of freedom for the complete density matrix. The trace over the momentum is obtained by the integration over the momentum which was considered ambiguous since there are momentum dependent components in the Dirac spinor. This ambiguity has been cleared by considering the problem in the FW representation which has been shown to be equivalent to the covariant representation. The spin

entropy which describes the purity of the spin changes under the Lorentz transformation. The pure spin state can be totally mixed spin state and vice versa under the Lorentz transformation. Therefore the entropy of the spin is not either Lorentz invariant nor covariant.

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